Lab #3 for Geophysical Inverse Theory ESS 523, Fall 2005, Univ. of WA.
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Today we're going to implement Ken's density-of-the Earth problem, using material from his lectures. We will estimate the density profile of the Earth given the two data points of mass and moment of inertia of the Earth, and also given one of several choices of parameterization of the problem.

This problem follows the Gram matrix approach seen in Parker's book. As detailed in Ken's lectures, our density profile problem is a linear forward problem in the form of a Fredholm Integral Equation of the First Kind, or "IFK":

$$d(s) = \int g(s,x)\, m(x)\, dx$$

continuous

In our case the $m(x)$ is density as a function of Earth radius, the integral is over Earth's radius, $d(s_1)$ and $d(s_2)$ are two discrete data points of Earth's mass and moment of inertia, and $g()$ is the physics that relate the density model to those data.

For a problem like this, Ken's lectures (and Parker's book) gave us a method using the Gram matrix to solve this problem for the $m(x)$ that both fits the data exactly, as per:

$$d_i = (g_i, m) \qquad i = 1 \dots N$$

note inner product notation

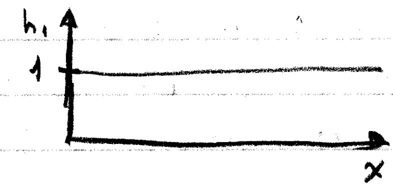
and also meets the following criteria that we choose and set up by design. Each of the three following cases is an example of one way to set up the problem given some prior knowledge about the Earth's structure. For this lab we will try all three and compare how they affect the resulting density profile.

1. $m(x)$ is the "smallest" model, ie has smallest model norm $\|m\| = (m,m)$ of all possible solutions.
2. $m(x)$ is the closest of all possible solutions to some preferred model $m^*(x)$, so that what's smallest is $\|m - m^*\|$.

→

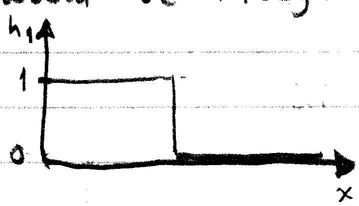
3. $m(x)$ is the smallest solution that doesn't penalize some properties of $m(x)$, which you can express in terms of linearly independent functions $h_k(x)$.
For example, maybe you want the mean subtracted off before finding the smallest $m(x)$ — then you'd just have a single $h_k(x) = h_1(x) = 1$, so $h_1(x)$ looks like:

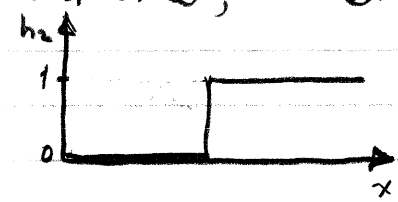
you'll use this in this lab assignment



If there were multiple regions in your x domain where you didn't want to penalize the mean of each region, then you could have one of these h_k's for each region, and they would be orthogonal to each other, for example:

you'll use these in this lab assignment





(Recall that as orthogonal functions these multiply together at each x and then sum over x to equal zero.)

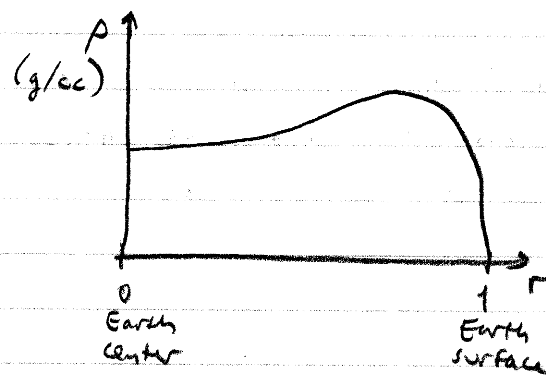
But 1, 2, and 3. are just example possibilities, and there are yet other ways one may want to set up the problem, including maybe some combination of 2. and 3..

Note that we've been ramping up our Matlab skills over the past few labs, using supplied scripts or cutting and pasting code. Now in this lab you'll write all your own code.

For this lab we are given the data and the g's, and for each of cases 1, 2, and 3 we wish to find the spherically symmetric density of the Earth as a function of depth.
So our model is $m(x) = \rho(r)$ ← radius

and we'll use normalized radius so it goes from 0 at the Earth's center to 1 at its surface: $r : 0 \rightarrow 1$.
For each of the three cases we'll plot the resulting density profile $\rho(r)$ so the plot looks something like this:

$m(x) = \rho(r)$



Our model $\rho(r)$ will be in the Hilbert space $\overset{\text{twice differentiable}}{L^2} \underbrace{[0, 1]}_{\text{domain of } r}$

So we have an inner product, and that's defined as:

$$(a, b) = \int_0^1 a(r)\, b(r)\, dr$$

Note our inner product integrates over the Earth's radius.

The vector of data points you'll use is:

$$\underline{d} = \begin{bmatrix} 1.839 \\ 0.9125 \end{bmatrix} \begin{array}{l} \leftarrow \text{related to mass of the Earth} \\ \leftarrow \text{related to moment of inertia of the Earth} \end{array}$$

(assuming spherically symmetric)

Next, and this is a key concept, this method will produce a continuous function for the density profile $\rho(r)$. But since we want to plot that function or list values of that function in a table, we must compute a series of ρ values from $\rho(r)$ at a series of r values that we choose.

So for this lab let's choose those r values in Matlab to be:

$$r = [0 : 0.05 : 1]'$$

(apos. makes it a column vector)

You could choose a different vector of r values here, and you would still be sampling from the same $\rho(r)$ function. But note that if you choose the r's <u>too</u> sparsely you may miss features in $\rho(r)$ you'd like to see.

So as derived in class lecture, our data kernel functions g_i will be r^2 and r^4, and these will form a matrix given the vector of r values we choose above:

$$g_i(r) \rightarrow \underline{\underline{g}} = [g_1(r) \;\; g_2(r)] = [\underline{r}^2 \;\; \underline{r}^4] = \begin{bmatrix} \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix}$$

oops, note notational confusion between the different quantities of <u>matrix</u> $\underline{\underline{g}}$ and <u>functions</u> $g_i(r)$

The Gram matrix for this problem was also derived in class:

$$\underline{\underline{\Gamma}} = \begin{bmatrix} 1/5 & 1/7 \\ 1/7 & 1/9 \end{bmatrix}$$

The whole point with using this Gram matrix business is parameterization of the problem, to turn a continuous problem into a discrete one. The reason is that IFK's can be really hard to solve directly for the continuous function $m(x)$, ie. $\rho(r)$ in this case.

So to this end we represent $\rho(r)$ as a sum of weighted basis functions. Those basis functions might include only the physics of the problem as in case 1., or as in case 3 additional basis functions to address the additional criteria we impose:

$$\rho(r) = \underbrace{\sum_{j=1}^{N} \alpha_j g_j(r)}_{\text{physics of the problem}} + \underbrace{\sum_{k=1}^{M} \beta_k h_k(r)}_{\text{additional criteria}}$$

case 1

case 3

(the focus of this lab is cases 1 and 3, with case 2 as extra credit. We'll discuss case 2 in a few more pages.)

If we solve for the α's and β's, we have everything needed to compute $\rho(r)$ at any r, because all the $g_j(r)$ and $h_k(r)$ are known. The Gram matrix $\underline{\Gamma}$ relates the α's to the data points in $\underline{d}$, and in case 3 we'll use another matrix $\underline{\underline{A}}$ that relates the β's to the problem.

The way the physics of the problem were defined, there are only two $g_j(r)$ so there are two α's in the vector $\underline{\alpha}$. Similarly, the Gram matrix $\underline{\Gamma}$ is 2×2. But the additional criteria can be defined however you like (well, as you think is appropriate for your problem, that is). So there can be any number of $h_k(r)$, and hence different dimensions to $\underline{\underline{A}}$. However, there are only two data points, so we'll see that as we add too many more parameters β_k you don't learn much about the model.

Case 1. is really easy to solve, because its criteria is already inherently part of the simple least-squares solution. Ie we know that

$$\underline{\alpha} = \underline{\underline{\Gamma}}^{-1} \underline{d}$$

will give us the $\underline{\alpha}$ that has the smallest model norm while fitting the data $\underline{d}$ exactly. (Note there is no noise on the data in this problem.) So we don't need any additional basis functions and β's for case 1.

Given the $\underline{\alpha}$ you compute as above, and the $\underline{\underline{g}}$ matrix from a few pages ago which contain the $g_j(r)$ for a set of r's, you can obtain your $\underline{\rho}$ values to plot (ie samples from $\rho(r)$):

$$\underline{\rho} = \underline{\underline{g}}\,\underline{\alpha} \qquad \left(\rho_i = \sum_{j=1}^{N} \alpha_j \, g_j(r_i) \right)$$

In Matlab you could plot using "plot(r, rho)"
Please label the axes and use a plot legend for all your results; you learned how to do that in earlier labs...

Let's concentrate on case 3. next, with two variations 3a and 3b. You approach them both the same but with some differences in the parameterization, and you'll see that case 1. was also really just a special case of these parameterizations.
We'll save case 2 for the ~~end~~ since that will be the extra credit case...

Let's define the criteria for cases 3a and 3b more specifically:

3a. we want a solution that doesn't penalize the mean value of $\rho(r)$ over the whole radius of the Earth. In other words, the mean ρ gets subtracted off before minimizing $\|\rho(r)\|$.

3b. we want a solution that considers the Earth in two layers instead of one, a mantle and a core, and that doesn't penalize the mean of the core when you're in the core, and doesn't penalize the mean of the mantle when you're in the mantle.

In both of these cases we have our $g()$ functions, ie $[g_1(r)\ g_2(r)] = [r^2\ r^4]$ and we choose an appropriate set of $h()$ functions, which are different for 3a and 3b. From these g and h functions we calculate our A_{ij} matrix by hand:

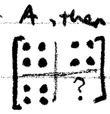
$$A_{ij} = (g_i, h_j) = \int_0^1 g_i(r)\, h_j(r)\, dr$$

And recall that we broke up our model $\rho(r)$ in two expansions, one based in the physics of the problem and the other based in the null space of the problem:

$$\rho(r) = \sum_{j=1}^{N} \alpha_j g_j(r) + \sum_{k=1}^{K} \beta_k h_k(r)$$

Once we find the α's and β's we can find our best $\rho(r)$ because we already know our h's and g's. And we saw in lecture that we can obtain our α's and β's like this:

Hint #1: how to figure out dimensions of the zero vector and matrix? You'll know dim. of A, then draw dots to see what elements are left ...



$$\begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \end{bmatrix} = \begin{bmatrix} \underline{\Gamma} & \underline{\underline{A}} \\ \underline{\underline{A}}^T & \underline{\underline{0}} \end{bmatrix}^{-1} \begin{bmatrix} \underline{d} \\ \underline{0} \end{bmatrix}$$

Hint #2: how to enter this block matrix into Matlab? Do something like:

[G A ; A' zeros(2)]

note this will look different for 3a and 3b

We did see in class other equivalent expressions for deriving $\underline{\alpha}$ and $\underline{\beta}$ separately. Myself I just find this formulation easier...

For case 3a., we don't want to penalize the mean over the whole radius of the Earth and at the beginning of this lab lecture we were given the way to do that, which is to have a single h function that is always 1: $h_1(r) = 1$.

So you can plug in this simple $h_1(r)$ along with $g_1(r)$ and $g_2(r)$ and do the integration in the formula for A_{ij}, where the i's and j's are either one or two.

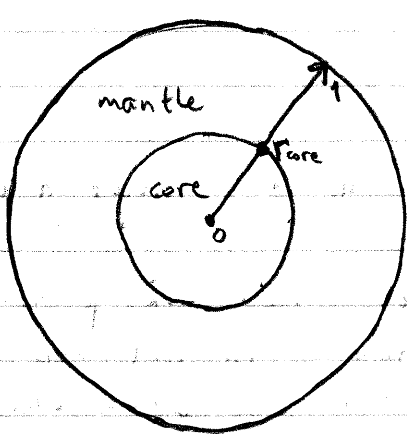
Note the dimension of $\underline{\underline{A}}$ is dependent on the number of g's and number of h's you have:

$$\underline{\underline{A}} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

and as a hint, this value should be $1/3$

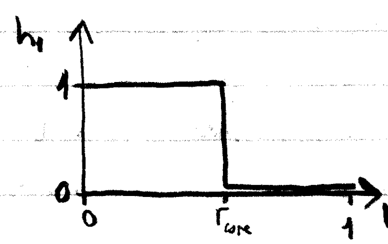
There. Now you have everything you need for 3a.

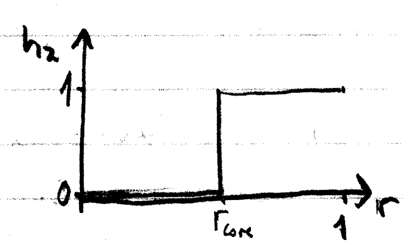
In 3b we have 2 layers, a core and a mantle, and the boundary between them is at r_{core}. Let us use $\boxed{r_{core} = 0.547.}$



Again, we wish to not penalize the mean density of the core while we're in the core, and to not penalize the mean density of the mantle while we're in the mantle.

So we choose $h_k(r)$'s as I alluded to earlier:





We create the A_{ij} matrix the same way as in 3a, but now the dimensions are different because we have a different number of h's:

$$\underline{\underline{A}} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

as a hint, this value should be $(1-r_{core}^3)/3$

as a hint, this value should be $r_{core}^5/5$

And that's everything you need to find $\rho(r)$ in 3b. Remember in all cases we plug the α's and β's back into the sums with the g's and h's to get $\rho(r)$.

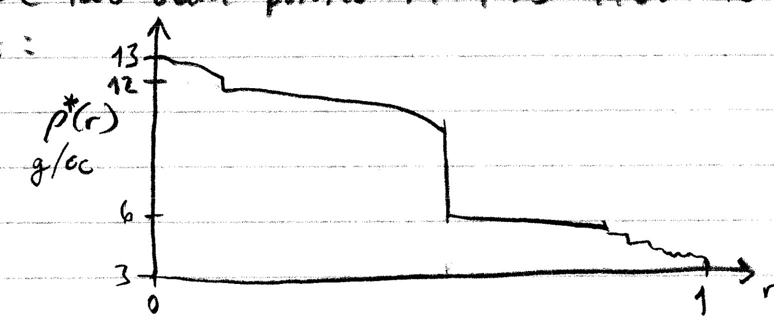
Okay, now computing these results for the different cases is one important part of this lab assignment, but of course the other is to understand what you've computed. So here are a few leading questions to that end, which you must answer as part of the assignment:

- Compare the plotted results of cases 1 and 3a. Do you think 1 could be the density profile of a real planet? Why not?
- You'll find that the $\rho(r)$ for 1 seems to "clamp down" at a certain value while $\rho(r)$ for 3a doesn't. What accounts for this difference?
- You'll find that $\rho(r)$ for 3b will look significantly different than that for 1 and 3a. The goal now is to understand what caused it to be so different. (First of all, what about the profile from 3b makes it unrealistic as a planetary density profile?) How many data points were there in each of 1, 3a, and 3b? How many parameters in each case being estimated from those points? Do some of the parameters end up zero? Which ones? What's going on?

Now that's more than enough for one lab assignment. But let's finish by discussing case 2, in case you want to add some extra credit to your lab, or just have these ideas a little more complete in your notes.

In case 2 we wish to find the $\rho(r)$ that is the closest of all possible solutions to some preferred model $\rho^*(r)$. So for example, maybe we have an existing density profile based on some other type of measurement, and for the parts of the model that our data points cannot constrain, we would like to revert to that existing profile.

One such profile you could use for this is the density profile within the Preliminary Earth Model (PREM), which comes from seismic measurements. It's pretty detailed compared to what you could pull out of the two data points in this lab. It looks something like this:



(PREM is readily available online. Try Google with "PREM earth model")

This problem is set up for you in Ken's notes on page 21, and he mentioned it in a class lecture as well.
Basically you just say $\delta m = m - m^*$, i.e. $\delta\rho = \rho(r) - \rho^*(r)$ and you redefine your data vector as $\tilde{d}_j = d_j - (g_j, m^*) = (g_j, \delta m)$.
So you solve for $\underline{\alpha} = \underline{\Gamma}^{-1}\underline{\tilde{d}}$ and then $\rho(r) = \sum_{j=1}^{\tilde{N}} \alpha_j g_j(r) + \rho^*(r)$

So that's it for this lab. Just remember lastly that one can combine these cases as well, for example maybe cases 2 and 3a.
See you next week!